

AN IMPROVED UPPER BOUND FOR THE ARGUMENT OF THE RIEMANN ZETA-FUNCTION ON THE CRITICAL LINE II

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ABSTRACT. This paper concerns the function $S(T)$, the argument of the Riemann zeta-function along the critical line. The main result is that

$$|S(T)| \leq 0.111 \log T + 0.275 \log \log T + 2.450,$$

which holds for all $T \geq e$.

1. SUMMARY OF RESULTS

This paper is the sequel to [14] and is related to [13]; reference will be made frequently to these papers. Write

$$(1.1) \quad |S(T)| \leq a \log T + b \log \log T + c, \quad \text{for } T \geq T_0,$$

whence the following table provides a brief historical summary.

TABLE 1. Bounds on $S(T)$ in (1.1)

	a	b	c	T_0
Von Mangoldt [15] 1905	0.432	1.917	12.204	28.588
Grossmann [5] 1913	0.291	1.787	6.137	50
Backlund [1] 1914	0.275	0.979	7.446	200
Backlund [2] 1918	0.137	0.443	4.35	200
Rosser [10] 1941	0.137	0.443	1.588	1467
Trudgian [14] 2012	0.17	0	1.998	e
Trudgian (Theorem 1) 2012	0.111	0.275	2.450	e

Note that the result in [14] improves on that in [10] when $25 \leq T \leq 10^{15}$. The purpose of this article is to improve on the result in [10] for all T . This is achieved with the following theorem.

Theorem 1.

$$|S(T)| \leq 0.111 \log T + 0.275 \log \log T + 2.450,$$

for $T \geq e$.

This implies the following result concerning $N(T)$, the number of complex zeroes of $\zeta(s)$ with imaginary parts in $(0, T)$.

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Corollary 1. For $T \geq T_0 \geq e$

$$\left| N(T) - \frac{T}{2\pi} \log \frac{T}{2\pi e} - \frac{7}{8} \right| \leq 0.111 \log T + 0.275 \log \log T + 2.450 + \frac{0.2}{T_0}.$$

It is known (see, e.g., [3, 11]) that¹

$$(1.2) \quad |S(T)| \leq 1, \text{ for } 0 \leq T \leq 280, \quad |S(T)| \leq 2, \text{ for } 0 \leq T \leq 6.8 \cdot 10^6.$$

The approach taken in this paper is to prove results initially for $T > T_0 > 6.8 \cdot 10^6$, and then for all T by using (1.2). Note that Theorem 1 is sharper than Rosser's bound in [10] whenever $T \geq 6.4 \cdot 10^6$; for smaller values of T one is better placed using (1.2), which is superior to both Theorem 1 and the bound in [10].

Explicit bounds on $S(T)$ are used in conjunction with the verification of the Riemann hypothesis to a certain height. Hence there is some interest in obtaining, not necessary the smallest coefficient of $\log T$ in Theorem 1, but good bounds of the form $|S(T)| \leq \alpha \log T$ for all $T \geq T_0$, where T_0 is the point up to which the Riemann hypothesis has been verified. From (1.1) one can write

$$|S(T)| \leq \log T \left(a + \frac{b \log \log T_0}{\log T_0} + \frac{c}{\log T_0} \right) = \alpha \log T,$$

for $T \geq T_0 \geq e^e$. Values of α have been provided in the following table. The parameters η and r are those found in (6.1).

TABLE 2. Bounds on $|S(T)| \leq \alpha \log T$ for $T \geq T_0$

T_0	α	η	r
10^6	0.260	0.28	2.28
10^7	0.246	0.24	2.35
10^8	0.235	0.22	2.38
10^9	0.226	0.19	2.44
10^{10}	0.218	0.17	2.49
10^{11}	0.212	0.16	2.51
10^{12}	0.206	0.15	2.53
10^{13}	0.202	0.14	2.51
10^{14}	0.197	0.13	2.50
10^{15}	0.193	0.12	2.48

Throughout the paper η denotes a parameter satisfying $0 < \eta \leq \frac{1}{2}$.

2. INTRODUCTION

Write $a(s) = (s-1)\zeta(s)$, whence $a(s)$ is an entire function. The function

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \frac{1}{2}s\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)a(s)$$

is entire and satisfies the functional equation

$$(2.1) \quad \xi(s) = \xi(1-s).$$

Let $N(T)$ denote the number of zeroes $\rho = \beta + i\gamma$ of $\zeta(s)$ for which $0 < \beta < 1$ and $0 < \gamma < T$. For any $\sigma_1 \in (1, 2]$ form a rectangle with vertices at $\sigma_1 \pm iT$ and

¹Indeed, (1.2) is the statement that Gram's Law holds for all $0 \leq T \leq 280$ and that Rosser's Rule holds for all $0 \leq T \leq 6.8 \cdot 10^6$.

$1 - \sigma_1 \pm iT$. Let \mathcal{C} denote the portion of the rectangle in the region $\Re(s) \geq \frac{1}{2}$ and $\Im(s) \geq 0$. Write \mathcal{C} as the union of two straight lines, viz. let $\mathcal{C} = \mathcal{C}_1 + \mathcal{C}_2$, where \mathcal{C}_1 connects 0 to $\sigma_1 + iT$ and \mathcal{C}_2 connects $\sigma_1 + iT$ to $\frac{1}{2} + iT$. From Cauchy's theorem and (2.1) one deduces that $\pi N(T) = \Delta_{\mathcal{C}} \arg \xi(s)$. Thus

$$(2.2) \quad \pi N(T) = \Delta_{\mathcal{C}} \arg \pi^{-s/2} + \Delta_{\mathcal{C}} \arg s(s-1) + \Delta_{\mathcal{C}} \arg \Gamma\left(\frac{s}{2}\right) + \pi S(T),$$

where

$$(2.3) \quad \pi S(T) = \Delta_{\mathcal{C}_1} \arg \zeta(s) + \Delta_{\mathcal{C}_2} \arg a(s) - \Delta_{\mathcal{C}_2} \arg(s-1).$$

The only terms in (2.2) and (2.3) that require more than a passing mention are $\Delta_{\mathcal{C}_1} \arg \zeta(s)$, $\Delta_{\mathcal{C}} \arg \Gamma\left(\frac{s}{2}\right)$ and $\Delta_{\mathcal{C}_2} \arg a(s)$. For the first use

$$|\arg \zeta(\sigma_1 + it)| \leq |\log \zeta(\sigma_1 + it)| \leq \log \zeta(\sigma_1),$$

and for the second use

$$\log \Gamma(z) = \left(z - \frac{1}{2}\right) \log z - z + \frac{1}{2} \log 2\pi + \frac{\theta}{6|z|},$$

(see, e.g., [7, p. 294]) which is valid for $|\arg z| \leq \frac{\pi}{2}$, and in which θ denotes a complex number satisfying $|\theta| \leq 1$. To estimate $\Delta_{\mathcal{C}_2} \arg a(s)$ write

$$(2.4) \quad f(s) = \frac{1}{2} \{a(s+iT)^N + a(s-iT)^N\},$$

for some positive integer N , to be determined later. Thus $f(\sigma) = \Re a(\sigma + iT)^N$. Suppose that there are n zeroes of $\Re a(\sigma + iT)^N$ for $\sigma \in \mathcal{C}_2$. These zeroes partition the segment into $n+1$ intervals. On each interval $\arg a(\sigma + iT)^N$ can increase by at most π , whence

$$|\Delta_{\mathcal{C}_2} \arg a(s)| = \frac{1}{N} |\Delta_{\mathcal{C}_2} \arg a(s)^N| \leq \frac{(n+1)\pi}{N}$$

In conclusion, when $T \geq 1$

$$(2.5) \quad \left| N(T) - \frac{T}{2\pi} \log \frac{T}{2\pi e} - \frac{7}{8} \right| \leq \frac{1}{4\pi} \tan^{-1} \frac{1}{2T} + \frac{T}{4\pi} \log \left(1 + \frac{1}{4T^2} \right) + \frac{1}{3\pi T} + |S(T)| \\ \leq \frac{0.2}{T} + |S(T)|,$$

where

$$|S(T)| \leq \frac{(n+1)}{N} + \frac{1}{\pi} \log \zeta(\sigma_1) + \frac{1}{\pi} \tan^{-1} \frac{1}{2T} + \frac{1}{\pi} \tan^{-1} \frac{\sigma_1}{T} \\ \leq \frac{(n+1)}{N} + \frac{1}{\pi} \log \zeta(\sigma_1) + \frac{0.8}{T}.$$

The inequality in (2.5) enables one to deduce Corollary 1 from Theorem 1.

3. ESTIMATING n

One may estimate n with Jensen's Formula.

Lemma 1 (Jensen's Formula). *Let $f(z)$ be holomorphic for $|z - a| \leq R$ and non-vanishing at $z = a$. Let the zeroes of $f(z)$ inside the circle be z_k , where $k = 1, 2, \dots, n$, and let $|z_k - a| = r_k$. Then*

$$(3.1) \quad \log \frac{R^n}{|r_1 r_2 \cdots r_n|} = \frac{1}{2\pi} \int_0^{2\pi} \log f(a + Re^{i\phi}) d\phi - \log |f(a)|.$$

The following lemma, proved in [13], is now used to invoke Backlund's trick. For a complex-valued function $f(s)$, define $\Delta_{\pm} \arg f(s)$ to be the change in argument of $f(s)$ as σ varies from $\frac{1}{2}$ to $\frac{1}{2} \pm \delta$, where $\delta > 0$.

Lemma 2. (i) *Let N be a positive integer and let $T \geq T_0 \geq 1$. Suppose that*

$$|\Delta_+ \arg a(s) + \Delta_- \arg a(s)| < E,$$

where $E = E(\delta, T_0)$. If there are n zeroes of $\Re a(\sigma + iT)^N$ for $\sigma \in [\frac{1}{2}, \sigma_1]$, then there are at least $n - 1 - [NE/\pi]$ zeroes in $\sigma \in [1 - \sigma_1, \frac{1}{2}]$.

(ii) *Denote the zeroes in $[\frac{1}{2}, \sigma_1]$ by $\rho_{\nu} = a_{\nu} + iT$ where $\frac{1}{2} \leq a_n \leq a_{n-1} \leq \cdots \leq \sigma_1$, and the zeroes in $[1 - \sigma_1, \frac{1}{2}]$ by $\rho'_{\nu} = a'_{\nu} + iT$ where $1 - \sigma_1 \leq a'_1 \leq a'_2 \leq \cdots \leq \frac{1}{2}$. Then*

$$a_{\nu} \geq 1 - a'_{\nu}, \quad \text{for } \nu = 1, 2, \dots, n - 1 - [NE/\pi],$$

and, if $\sigma_1 = \frac{1}{2} + \sqrt{2}(\eta + \frac{1}{2})$, then

$$(3.2) \quad \prod_{\nu=1}^n |1 + \eta - a_{\nu}| \prod_{\nu=1}^{n-1-[NE/\pi]} |1 + \eta - a'_{\nu}| \leq (\frac{1}{2} + \eta)^{2n-1-[NE/\pi]}.$$

Proof. This was proved in [13, Lemma 2] for Dirichlet L -functions; the proof for $a(s) = (s-1)\zeta(s)$ is identical. \square

3.1. Calculation of E . One may use [13, (5.4)] to estimate E in Lemma 2.

$$\begin{aligned} E \leq G(\delta, T) = & \left(-\frac{5}{4} + \frac{\delta}{2}\right) \tan^{-1} \frac{\frac{1}{2} + \delta}{T} - \left(\frac{5}{4} + \frac{\delta}{2}\right) \tan^{-1} \frac{\frac{1}{2} - \delta}{T} + \frac{5}{4} \tan^{-1} \frac{1}{2T} \\ & - \frac{T}{4} \log \left[1 + \frac{2\delta^2(T^2 - \frac{1}{4}) + \delta^4}{(T^2 + \frac{1}{4})^2} \right] + \frac{4\theta}{3T}. \end{aligned}$$

One can show that $G(\delta, T)$ is decreasing in T and increasing in δ . Therefore, since, in Lemma 2 (i) one takes $\sigma_1 = \frac{1}{2} + \sqrt{2}(\frac{1}{2} + \eta)$, it follows that $\delta \leq \sqrt{2}(\frac{1}{2} + \eta)$, whence

$$E \leq G(\sqrt{2}, T_0) \leq \frac{4.4}{T_0},$$

for $T \geq T_0$.

3.2. Applying Jensen's Formula. In Lemma 1, take $a = 1 + \eta$, $f(z)$ as in (2.4), and $R = r(\frac{1}{2} + \eta)$, where $r > 1$. Suppose that there are n zeroes of $\Re a(\sigma + iT)^N$ for $\sigma \in [\frac{1}{2}, \sigma_1]$, where $\sigma_1 = \frac{1}{2} + \sqrt{2}(\eta + \frac{1}{2})$. Initially, to take advantage of Backlund's trick, one needs $1 + \eta - r(\frac{1}{2} + \eta) \leq 1 - \sigma_1$, so that all of the zeroes are included in the contour. The argument in [13, §4.1] shows that one can use any $r > 1$. The following results are simplified greatly if one imposes an upper bound on r . Indeed, to use (4.8) requires $r(\frac{1}{2} + \eta) \leq \frac{3}{2} + \eta \leq 2$.

To apply Jensen's formula it is necessary to show that $f(1 + \eta)$ is non-zero: this is easy to do upon invoking an observation due to Rosser [10]. Write $a(1 + \eta + iT) =$

$Ke^{i\psi}$, where $K > 0$. Choose a sequence of N 's tending to infinity for which $N\psi$ tends to zero modulo 2π . Thus

$$(3.3) \quad \frac{f(1+\eta)}{|a(1+\eta+iT)|^N} \rightarrow 1.$$

It follows from (3.1) and (3.2) that

$$n \leq \frac{1}{4\pi \log r} J - \frac{1}{2 \log r} \log |f(1+\eta)| + \frac{1}{2} + \frac{NE}{2\pi},$$

where

$$J = \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \log |f(1+\eta+r(\frac{1}{2}+\eta)e^{i\phi})| d\phi.$$

First one may bound $\log |f(1+\eta)|$ using (3.3) and the trivial bound $|\zeta(s)| \geq \frac{\zeta(2\sigma)}{\zeta(\sigma)}$. Thus

$$\log |f(1+\eta)| = N \log |a(1+\eta+iT)| + o(1) \geq N \left(\log T + \log \frac{\zeta(2+2\eta)}{\zeta(1+\eta)} \right) + o(1).$$

4. ESTIMATING THE INTEGRALS

Divide J into five pieces thus

$$R_0 = \{s : 1+\eta \leq \sigma \leq 1+\eta+r(\frac{1}{2}+\eta)\} = \{\phi : -\frac{\pi}{2} \leq \phi \leq \frac{\pi}{2}\},$$

$$R_1 = \{s : 1 \leq \sigma \leq 1+\eta \text{ and } t \geq T\} = \{\phi : \frac{\pi}{2} \leq \phi \leq \frac{\pi}{2} + \phi_1\},$$

$$R_2 = \{s : \frac{1}{2} \leq \sigma \leq 1 \text{ and } t \geq T\} = \{\phi : \frac{\pi}{2} + \phi_1 \leq \phi \leq \frac{\pi}{2} + \phi_2\},$$

$$R_3 = \{s : 0 \leq \sigma \leq \frac{1}{2} \text{ and } t \geq T\} = \{\phi : \frac{\pi}{2} + \phi_2 \leq \phi \leq \frac{\pi}{2} + \phi_3\},$$

$$R_4 = \{s : 1+\eta-r(\frac{1}{2}+\eta) \leq \sigma \leq 0 \text{ and } t \geq T\} = \{\phi : \frac{\pi}{2} + \phi_3 \leq \phi \leq \pi\},$$

where

$$\phi_1 = \sin^{-1} \frac{\eta}{r(\frac{1}{2}+\eta)}, \quad \phi_2 = \sin^{-1} \frac{1}{r}, \quad \phi_3 = \sin^{-1} \frac{1+\eta}{r(\frac{1}{2}+\eta)}.$$

One may then write $J = \int_{R_0} + 2 \int_{R_1} + 2 \int_{R_2} + 2 \int_{R_3} + 2 \int_{R_4}$. In estimating each integral some small error terms labelled $\epsilon_0, \dots, \epsilon_4$ are encountered. Since these are all $O(T_0^{-1})$, they have been estimated with a great deal of alacrity. It is neither essential nor insightful to strive for the smallest bounds on these terms.

4.1. Convexity bounds. To estimate $\zeta(s)$ on R_0 one may use the trivial estimate $|\zeta(s)| \leq \zeta(\sigma)$. On R_1, \dots, R_4 one can use the following version of the Phragmén–Lindelöf principle.

Lemma 3. *Let a, b, Q and k be real numbers, such that $Q+a > 1$, and let $f(s)$ be regular analytic in the strip $a \leq \sigma \leq b$ and satisfy the growth condition*

$$|f(s)| < C \exp \left\{ e^{k|t|} \right\},$$

for a certain $C > 0$ and for $0 < k < \pi/(b-a)$. Also assume that

$$|f(s)| \leq \begin{cases} A|Q+s|^{\alpha_1} (\log |Q+s|)^{\alpha_2} & \text{for } \Re(s) = a, \\ B|Q+s|^{\beta_1} (\log |Q+s|)^{\beta_2} & \text{for } \Re(s) = b, \end{cases}$$

where $\alpha_1 \geq \beta_1$ and where $\alpha_1, \alpha_2, \beta_1, \beta_2 \geq 0$. Then throughout the strip $a \leq \sigma \leq b$ the following holds

$$|f(s)| \leq \{A|Q+s|^{\alpha_1} |\log(Q+s)|^{\alpha_2}\}^{\frac{b-\sigma}{b-a}} \{B|Q+s|^{\beta_1} |\log(Q+s)|^{\beta_2}\}^{\frac{\sigma-a}{b-a}}.$$

Proof. This extends a result due to Rademacher [9, pp. 66-67] so as to incorporate logarithms. Form the function

$$F(s) = f(s)\phi(s; Q)E^{-1}e^{-vs} \{\log(Q+s)\}^{\frac{\alpha_2(\sigma-b)+\beta_2(a-\sigma)}{b-a}},$$

where $\phi(s; Q)$ is the function of [9, Theorem 1], and E and ν are determined by $A = Ee^{\nu a}$ and² $B = Ee^{\nu b}$. Since $Q+a > 1$, the function $F(s)$ is holomorphic in the strip $a \leq \sigma \leq b$. The proof now proceeds as in [9]. \square

Lemma 3 will be applied to R_1, \dots, R_4 where it will be convenient to write $|Q+s|$ in terms of T . If $Q = Q_0 \leq 1000$ and $T \geq T_0 \geq 10^6$ one may write

$$(4.1) \quad |\log |Q_0 + s| - \log T| \leq \frac{6}{T_0} \leq 10^{-5}.$$

If, in addition, $Q_0 \geq 1$, then $|\arg(Q_0 + s)| \leq \frac{\pi}{2}$ on R_1, \dots, R_4 . Using this, (4.1), and the identity

$$|\log z| = |\log |z|| \left\{ 1 + \left(\frac{\arg z}{\log |z|} \right)^2 \right\}^{1/2},$$

one deduces that

$$(4.2) \quad |\log(Q_0 + s)| \leq 1.007 \log T, \quad \log |\log(Q_0 + s)| \leq \log \log T + 0.007.$$

4.2. R_0 . On R_0

$$|a(s)| \leq |\eta + r(\frac{1}{2} + \eta)e^{i\phi} \pm iT| \zeta(1 + \eta + r(\frac{1}{2} + \eta) \cos \phi),$$

whence

$$\frac{R_0}{N} \leq \pi(\log T + \epsilon_0) + \int_{-\pi/2}^{\pi/2} \log \zeta(1 + \eta + r(\frac{1}{2} + \eta) \cos \phi) d\phi,$$

where

$$\epsilon_0 = \frac{r(\frac{1}{2} + \eta)}{T_0} + \frac{\{\eta + r(\frac{1}{2} + \eta)\}^2}{2T_0^2} \leq \frac{2}{T_0} + \frac{25}{8T_0^2} \leq \frac{3}{T_0}.$$

4.3. R_1 . On $\sigma = 1 + \eta$ bound $\zeta(\sigma)$ trivially, whence, for any $Q_0 \geq 0$

$$(4.3) \quad |a(1 + \eta + it)| \leq |Q_0 + (1 + \eta + it)| \zeta(1 + \eta).$$

On $\sigma = 1$ one may make use of Backlund's estimate [2, (53)] that

$$(4.4) \quad |\zeta(1 + it)| \leq \log t,$$

for $t \geq 50$. This, and a computation check for small t shows that

$$(4.5) \quad |a(1 + it)| \leq |Q_0 + (1 + it)| \log |Q_0 + (1 + it)|,$$

for all t and for any $Q_0 \geq 1$. It follows from Lemma 3, (4.1), (4.2), (4.3) and (4.5) that

$$\begin{aligned} \frac{R_1}{N} &\leq (\log T + 10^{-5})\phi_1 + \log \zeta(1 + \eta) \left[\phi_1 - \frac{r(\frac{1}{2} + \eta)(1 - \cos \phi_1)}{\eta} \right] \\ &\quad + (\log \log T + 0.007) \left[\frac{r(\frac{1}{2} + \eta)(1 - \cos \phi_1)}{\eta} \right]. \end{aligned}$$

²Note that in [9, (3.7)] there is a typo: $B = e^{\nu b}$ should read $B = Ee^{\nu b}$.

4.4. R_2 . Suppose that one is equipped with a bound

$$|\zeta(\frac{1}{2} + it)| \leq k_1 t^{k_2} (\log t)^{k_3}, \quad \text{for } t \geq t_0,$$

in which $0 \leq k_3 \leq 10$, say. This upper bound on k_3 is imposed merely to simplify the resulting error term. The convexity bound for $\zeta(s)$ shows that $k_2 \leq \frac{1}{4}$.

It follows that

$$(4.6) \quad |a(\frac{1}{2} + it)| \leq k_1 |Q_0 + (\frac{1}{2} + it)|^{k_2+1} (\log |Q_0 + (\frac{1}{2} + it)|)^{k_3}, \quad \text{for } t \geq t_0,$$

for any $Q_0 \geq 0$. It is always possible to choose Q_0 large enough so that (4.6) holds for all t . It follows from Lemma 3, (4.1), (4.2), (4.5) and (4.6) that

$$\begin{aligned} \frac{R_2}{N} &\leq (\log T + 10^{-5}) [2k_2 r(\frac{1}{2} + \eta)(\cos \phi_1 - \cos \phi_2) + (\phi_2 - \phi_1)(1 - 2k_2 \eta)] \\ &\quad + 2 \log k_1 [r(\frac{1}{2} + \eta)(\cos \phi_1 - \cos \phi_2) - \eta(\phi_2 - \phi_1)] \\ &\quad + 2(\log \log T + 0.007)[r(\frac{1}{2} + \eta)(1 - k_3)(\cos \phi_2 - \cos \phi_1) \\ &\quad \quad + (\phi_2 - \phi_1)(\frac{1}{2} + \eta - k_3 \eta)]. \end{aligned}$$

4.5. R_3 . The functional equation

$$(4.7) \quad \zeta(s) = \pi^{s-\frac{1}{2}} \frac{\Gamma(\frac{1}{2} - \frac{1}{2}s)}{\Gamma(\frac{1}{2}s)} \zeta(1-s),$$

(see, e.g., [12, Ch. II]), the estimate

$$(4.8) \quad \left| \frac{\Gamma(\frac{1}{2} - \frac{1}{2}s)}{\Gamma(\frac{1}{2}s)} \right| \leq \left(\frac{|1+s|}{2} \right)^{\frac{1}{2}-\sigma},$$

for $-\frac{1}{2} \leq \sigma \leq \frac{1}{2}$ (see [8, p. 197]), and Backlund's estimate (4.4) show that

$$|\zeta(it)| \leq \left(\frac{|1+it|}{2\pi} \right)^{\frac{1}{2}} \log t, \quad \text{for } t \geq 50.$$

A computational check shows that

$$(4.9) \quad |a(it)| \leq (2\pi)^{-\frac{1}{2}} |Q_0 + it|^{\frac{3}{2}} \log |Q_0 + it|,$$

for all t and for any $Q_0 \geq 2$. It follows from Lemma 3, (4.1), (4.2), (4.6) and (4.9) that

$$\begin{aligned} \frac{R_3}{N} &\leq (\log T + 10^{-5}) [r(\frac{1}{2} + \eta)(1 - 2k_2)(\cos \phi_2 - \cos \phi_3) \\ &\quad + (\phi_3 - \phi_2)\{2k_2(1 + \eta) + \frac{1}{2} - \eta\}] \\ &\quad + 2(\log \log T + 0.007) [(k_3 - 1)r(\frac{1}{2} + \eta)(\cos \phi_3 - \cos \phi_2) \\ &\quad \quad + (\phi_3 - \phi_2)\{k_3(1 + \eta) - (\frac{1}{2} + \eta)\}] \\ &\quad + \log 2\pi [(\frac{1}{2} + \eta)(\phi_3 - \phi_2) - r(\frac{1}{2} + \eta)(\cos \phi_2 - \cos \phi_3)] \\ &\quad + 2 \log k_1 [(1 + \eta)(\phi_3 - \phi_2) - r(\frac{1}{2} + \eta)(\cos \phi_2 - \cos \phi_3)]. \end{aligned}$$

4.6. R_4 . (4.7), (4.8), and the trivial estimate for $\zeta(s)$ show that

$$(4.10) \quad |a(q+it)| \leq (2\pi)^{q-\frac{1}{2}} |Q_0 + (q+it)|^{\frac{3}{2}-q} \zeta(1-q),$$

for all t and for all $Q_0 \geq 2$, where $q = 1 + \eta - r(\frac{1}{2} + \eta)$.

Lemma 3, (4.1), (4.2), (4.9) and (4.10) show that

$$\begin{aligned} \frac{R_4}{N} &\leq (\log T + 10^{-5})[(\tfrac{1}{2} - \eta)(\tfrac{\pi}{2} - \phi_3) + r(\tfrac{1}{2} + \eta)] \\ &\quad + (\log \log T + 0.007) \left[\frac{r(\tfrac{1}{2} + \eta)(\tfrac{\pi}{2} - \phi_3 - \cos \phi_3)}{r(\tfrac{1}{2} + \eta) - (1 + \eta)} \right] \\ &\quad + \log 2\pi[(\tfrac{1}{2} + \eta)(\tfrac{\pi}{2} - \phi_3) - r(\tfrac{1}{2} + \eta) \cos \phi_3] \\ &\quad + \log \zeta(r(\tfrac{1}{2} + \eta) - \eta) \left[\frac{r(\tfrac{1}{2} + \eta) - (\tfrac{\pi}{2} - \phi_3)(1 + \eta)}{r(\tfrac{1}{2} + \eta) - (1 + \eta)} \right]. \end{aligned}$$

One therefore has all of the results needed to bound $|S(t)|$. This produces an expression the inelegance of which prohibits its being inserted in this paper. The next section will provide some specific information.

5. SPECIFIC VALUES OF k_1, k_2 AND k_3

Cheng and Graham [4, Thm. 3] proved that

$$(5.1) \quad |\zeta(\tfrac{1}{2} + it)| \leq 1.457t^{1/6} \log t + 40.995t^{1/6} + 1.863 \log t + 123.125,$$

and that

$$(5.2) \quad |\zeta(\tfrac{1}{2} + it)| \leq 6t^{1/4} + 41.129,$$

where both (5.1) and (5.2) are valid for $t \geq e$. They combine these results with a computational check to show that

$$(5.3) \quad |\zeta(\tfrac{1}{2} + it)| \leq 3t^{1/6} \log t,$$

for $t \geq e$. Actually, their proof enables one to write 2.76 in place of 3 in (5.3). This can be improved by combining (5.1) not with (5.2) but with the estimate

$$|\zeta(\tfrac{1}{2} + it)| \leq \frac{4}{(2\pi)^{\frac{1}{4}}} t^{1/4},$$

(see, e.g., [6, Lemma 2]) which is valid for $t \geq 1$. This shows that

$$|\zeta(\tfrac{1}{2} + it)| \leq 2.38t^{1/6} \log t,$$

for $t \geq e$. This, and a small computational check for $0 \leq t \leq e$, shows that (4.6) holds with $k_1 = 2.38$, $k_2 = \frac{1}{6}$, $k_3 = 1$ and $Q_0 \geq 1$.

One can now take $Q_0 = 2$, whence all of (4.3), (4.5), (4.6), (4.9), (4.10) are satisfied.

6. CONCLUSION

Combining the above results shows that, when $T \geq T_0 \geq 1$,

$$(6.1) \quad |S(T)| \leq a \log T + b \log \log T + c,$$

where

$$a = \frac{\phi_1 \eta + (\phi_2 - \frac{3\pi}{2})(\tfrac{1}{2} + \eta) + \phi_3(1 + \eta) + r(\tfrac{1}{2} + \eta)(\cos \phi_1 + \cos \phi_2 + \cos \phi_3)}{6\pi \log r},$$

$$b = \frac{-\phi_1 + \phi_3 + r(\frac{1}{2} + \eta) \left\{ \frac{1 - \cos \phi_1}{\eta} + \frac{\frac{\pi}{2} - \cos \phi_3 - \phi_3}{r(\frac{1}{2} + \eta) - (1 + \eta)} \right\}}{2\pi \log r},$$

$$c = \frac{\log \frac{\zeta(1+\eta)}{\zeta(2+2\eta)}}{2 \log r} + \frac{1}{\pi} \log \zeta(\frac{1}{2} + \sqrt{2}(\eta + \frac{1}{2})) + \frac{\int_{-\pi/2}^{\pi/2} \log \zeta(1 + \eta + r(\frac{1}{2} + \eta) \cos \phi) d\phi}{4\pi \log r}$$

$$+ \left\{ \log \zeta(1 + \eta) \left[\phi_1 + \frac{r(\frac{1}{2} + \eta)(\cos \phi_1 - 1)}{\eta} \right] + (\frac{1}{2} + \eta)(\frac{\pi}{2} - \phi_2 - r \cos \phi_2) \log 2\pi \right.$$

$$- 2 \log k_1 [r(\frac{1}{2} + \eta)(2 \cos \phi_2 - \cos \phi_1 - \cos \phi_3) + \phi_2 - \phi_3 + \eta(2\phi_2 - \phi_1 - \phi_3)]$$

$$\left. + \left[\frac{(1 + \eta)(\frac{\pi}{2} - \phi_3) - r(\frac{1}{2} + \eta) \cos \phi_3}{1 + \eta - r(\frac{1}{2} + \eta)} \right] \log \zeta[r(\frac{1}{2} + \eta) - \eta] \right\} / (2\pi \log r) + 0.003.$$

Taking $\eta = 0.06$ and $r = 2.08$ proves Theorem 1 for $T \geq 6.8 \cdot 10^6$. When $T < 6.8 \cdot 10^6$, Theorem 1 follows from (1.2).

7. IMPROVEMENTS

Theorem 1 is improved instantly if one can provide better bounds for the growth of $|\zeta(\frac{1}{2} + it)|$ and $|\zeta(1 + it)|$. For the former one seeks an explicit bound on

$$(7.1) \quad |\zeta(\frac{1}{2} + it)| \ll t^\theta,$$

for some $\theta \leq \frac{1}{6}$. The easiest approach seems to be to make explicit the result of Titchmarsh [12, Thm. 5.18] which has $\theta = \frac{27}{164}$ in (7.1). For the latter one could make explicit the estimate $\zeta(1 + it) = O(\log t / \log \log t)$, by following the arguments preparatory to proving Theorem 5.16 in [12].

Both of these approaches are being investigated by the author.

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